gm°C and linear coefficient of thermal expansion 7.2×10^{-6} cm/cm °C. The resulting Grueneisen parameter is 0.50. WONDY III calculations were made using the above material constants and the experimental geometry; these results are shown in Fig. 3. The stress history for the completely developed wave in the absorbing specimen and the stress history at the front of the quartz gauge are both shown as well as the simple elastic response for instantaneous energy deposition. All theoretical response was calculated to give a peak propagating stress for instantaneous deposition 0.5 kilobar.

Discussion

Figure 3 shows the good agreement between the predicted and measured quartz response. The effect of the finite deposition time is very evident with the reduction and rounding of peak stresses and the increase in the time over which the stress reversal occurs. The effect of impedance mismatch is to reduce the peak stresses and further extend the time for stress reversal.

The experimental record plotted in Fig. 3 is that shown in Fig. 2 scaled to be equivalent to an instantaneous deposition stress of 0.5 kilobar. The actual peak stress for this shot was 0.116 kilobar, for 6.55 joules deposited over the 1.90 cm diameter spot. The measured stress was found to be linear with total deposited energy. If the fluence, joules/unit area, were reduced by a factor of 10, the experimental response was still within 10% of the predicted response. Shots taken at the same fluence level and during the same setup were identical with slight variations, less than 5%, in the stress levels. Similar variations were evident in the energy measurements. At no time did the experimental record vary more than 15% from the predicted response.

It is felt that the three major sources of error in the investigation were in the determination of deposited energy, the pressure measurements and the proper consideration of the effect of the bond thickness. The energy determination was probably made to within ±5%. The pressure measurements, including the errors in reading the traces had less than 10% error. Variations in the bond thickness should not have changed the results by more than 7%. The total experimental error should not have exceeded 20%.

The results of this study show that measurements of thermally generated elastic waves can be made which agree with the theories to well within the normal experimental error. It is questionable that the experimental error can be significantly reduced using the current techniques.

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Reduced-Order Modeling in Aircraft Mission Analysis

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A PARTICLE-dynamics model comprising three velocity and three position components is often used for aircraft motion; however, even simpler models are sometimes of value, e.g., the "energy state" model in which vertical motions are presumed executed instantaneously.[1,2] For some purposes, even lower order models as, for example, constant speed are of interest.[3,4] These models are of order six, four, and three, respectively, exclusive of integrations of variable mass. The present note derives an alternate third-order model featuring instantaneously variable speed by means of time-scale separation, further pursuing the approach taken in Refs. 1 and 2.

Energy State Model

The equations of state in energy approximation are, from Ref. 2:

$$\epsilon \dot{E} = V(T - D)/W \tag{1}$$

$$\dot{\chi} = gL \sin\mu/VW \tag{2}$$

$$\dot{x} = V \sin\chi \tag{3}$$

$$\dot{y} = V \cos\chi \tag{4}$$

The state variables are specific energy, $E = h + V^2/2g$, heading angle χ, and rectangular coordinates in the horizontal plane, x and y. The symbol V (velocity) is to be regarded as merely shorthand for $V = [2g(E - h)]^{1/2}$. Control variables are altitude h, bank angle μ, and a throttle variable η appearing in engine thrust T and in aerodynamic drag D (to account for speedbrake control). Drag is evaluated for level turning flight, $L \cos\mu = W$, i.e., this determines angle of attack. Weight W is taken constant for present purposes.

Instantaneously Variable Speed Model

The parameter ϵ appearing on the left of Eq. (1) is unity in the energy state approximation; it is introduced here preparatory to further reduction in order. The Tihonov time-scale separation technique, consisting of progressive reductions in order, was reviewed and applied to aircraft flight in Ref. 2. In that development, heading changes and energy changes were presumed to take place on the same time scale in the second of three groupings. Here, as a gross approximation, energy changes will be idealized as fast compared to heading changes. One sets $\epsilon = 0$, obtaining a reduced-order model, and solves the path-optimization problem (or whatever problem) with the end conditions involving energy E deleted, then accounts for energy transition to specified initial and terminal values in boundary-layer approximation, i.e., by local corrections calculated as afterthoughts. Boundary-layer corrections in optimization problems are dealt with in Refs. 2 and 5. Attention will focus in the following on the "reduced system" obtained from Eqs. (1–4) for $\epsilon = 0$.

With vanishing ϵ, the condition $V(T - D) = 0$ represents a constraint on the four control variables h, μ, η, E. Note that the variable E, and hence speed V, can now jump! For aircraft whose maximum level flight speeds are determined by available power and not by Mach number flight placard, two

Received August 21, 1970; revision received October 23, 1970.

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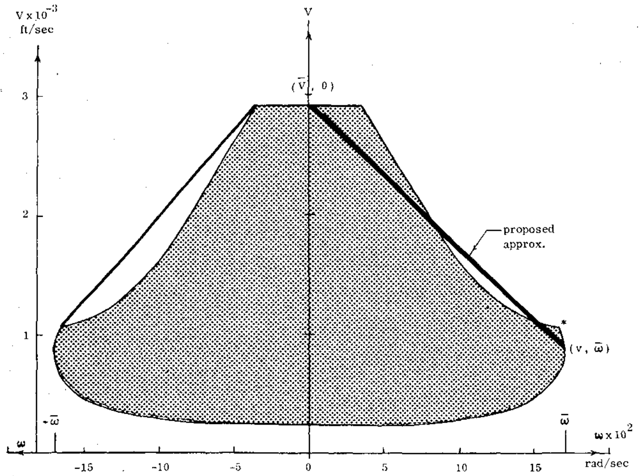

Fig. 1 Velocity-turn rate hodograph.

control-variable combinations are of particular interest: the maximum steady-speed point and the point of greatest level turning rate at which energy can be maintained. This last will usually be attained in high-lift-coefficient, low-altitude flight. It may correspond to maximum lift coefficient or maximum load factor, or it may be an "interior" extremum, depending upon the configuration, propulsion system, etc.; for a turbojet, this best turning circle will generally occur at sea level subsonically or transonically. It is associated with the classic circling encounter, *la ronde Lufbéry*, which is advantageous to the combatant with superior sustained turning performance, whether pursuer or evader.

Consider the model given by:

$$\dot{\chi} = \bar{\omega}\sigma; \quad -1 \leq \sigma \leq 1 \tag{5}$$

$$\dot{x} = V(\sigma) \sin\chi \tag{6}$$

$$\dot{y} = V(\sigma) \cos\chi \tag{7}$$

If $V(\sigma)$ is represented by

$$V(\sigma) = \bar{V} - (\bar{V} - v)|\sigma| \tag{8}$$

this corresponds to linear interpolation with respect to turn rate between maximum level flight speed $\bar{V}$ and the speed v at maximum turning rate $\bar{\omega}$. In some cases, it might roughly approximate part of the convex hull of a "relaxed" variational problem,[6] but perhaps it might best be regarded merely as a sweeping assumption, not too inconsistent with the already sweeping assumption of instantaneous jumps in speed.

The V, ω hodograph figure for the Mach 3 aircraft of Ref. 7 is illustrated in Fig. 1. It happens that, for this particular aircraft/propulsion system combination, turning rate is limited by the power available to maintain energy at almost all airspeed-altitude combinations rather than by normal load factor or lift-coefficient limits. An overpressure constraint limits altitude choice at speeds above the "corner" marked with an asterisk. The vee-shaped approximation of Eqs. (5) and (8) (right side of figure) is seen to bear some resemblance to the convex hull (left side). The tailoring of the approximation neglects the possibility of flight at low speeds $V < v$ which is rarely of interest in mission-shaping applications. The nonconvexity of the approximation itself also rules out its use where low-speed arcs enter the solution.

Optimal Reduced-Order Flight

It is not difficult to show that optimal flight, with the characteristic Eq. (8), takes place along arcs $\sigma = 1$, $\sigma = 0$, $\sigma = -1$ with no intermediate σ arcs. Sequences of the turn-dash-turn type are suggested by further reduction, i.e., by treating turns as initial and terminal boundary layers. Sectionally linear $V(\sigma)$ models offer obvious advantages for aerial combat analyses: simple performance characterization in terms of three turn-and-dash parameters, plus paths represented by a neat, predictable sequence of circular-arc turns and straight-line dashes. The attendant drawbacks are the unrealistic velocity jumps and their appearance suggests that the model is quite an approximate one.

Use of a model employing the $V(\sigma)$ of the actual hodograph figure, rather than the vee-shaped approximation, immediately raises the question of whether to relax or not, i.e., whether to accept "weak" extrema for the reduced-order problem or to adopt a $V(\sigma)$ characteristic employing the convex hull. This last option stretches the assumption of fast energy transitions in that dubious approximation of the "chattering" operation along the linear segment of the hull is implied. These considerations suggest that analyses proposing to deal with details of the hodograph figure's shape might better revert to a fourth-order energy model instead.

Pursuit/Evasion

It is of interest to consider the pursuit/evasion problem as a candidate for reduced-order approximation offered by third-order, variable-speed models. It may be recalled that with constant-speed models, the "homicidal chauffeur" game,[3,4] in which the evader may turn instantaneously, exhibits solutions dividing roughly into rectilinear tail-chases and "sidestepping" maneuvers, depending upon initial geometry. Solutions of the "game of two cars" with constant-speed models divide similarly, with sidestepping of diminished importance as the evader's turn performance is restricted. A version of this game with variable-speed models might be particularly interesting if an overlay of the evader's hodograph figure upon the pursuer's reveals a turn-rate superiority at any speed, such a choice of speed then being conducive to opening rather than closing the gap during a tail-chase. Qualitative insight into pursuit/evasion phenomena with variable-speed, third-order models could perhaps be gained by analyses of the two-car game employing the vee-shaped approximation.

Conclusion

The simplified models introduced in the preceding are not seriously proposed as competitors to energy modeling. They are advanced merely as slightly more realistic alternatives to a constant-speed model, for analyses in which it is important to hold the order to three on account of other complicating factors, e.g., multiple aircraft. While there is little really novel about piecing together missions with straight-line dashes and constant-radius turn arcs, perhaps reduced-order modeling is of some interest in furnishing a rationale for such a process. Third-order models appear to be of interest for use in pursuit/evasion analyses, extended versions of the game of two cars.

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